STUDY OF TRANSIENT HEAT TRANSFER IN THE CUTTING OF MATERIALS

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A general scheme is proposed for the solution of transient heat-transfer problems relating to the cutting (machining) of materials, due allowance being made for the interactions between the cutter, the work-piece, and the shavings.

Hardly any consideration has yet been given in the literature to the transient heat-transfer processes associated with the cutting of materials making due allowance for the mutual influence of all the participating components. Only steady-state heat transfer was considered in the best known monograph on the subject [1].

We shall here consider heat-transfer problems associated with cutting on the basis of a three-body system: cutter, work-piece, and shavings. The indices 1, 2, and 3 will be used to identify the thermophysical and geometrical coefficients of these constituents. Several contact surfaces s_i, $i = 1, 2, 3$, small by comparison with the dimensions of the actual bodies, take part in the thermal interaction. Friction releases a certain amount of heat $Q(t)$ in the overall contact region s; we denote the proportions absorbed by each body as $Q_i(t) = s_i q_i(t)$, where $q_i(t) = (1/s_i)Q_i(t)$ are the average thermal fluxes through the contact areas s_i. Within the limits of each contact area (by virtue of its small dimensions) we neglect the nonuniformity of the thermal flux, i.e., we envisage uniformly distributed heat sources with an intensity $q_i(t)$.

Let us consider that on the contact areas the following heat-balance equation is satisfied:

$$Q(t) = \sum_{i=1}^{3} s_i q_i(t). \tag{1}$$

The temperature at the tip of the cutter T(t) is common to all the bodies taking part in the cutting, and we call this the contact cutting temperature.

In the transient mode the contact temperature for each body may be expressed in the form of the thermal potential of a simple layer [2],

$$T(t) = \int_0^t q_i(t_0)\, J_i(t - t_0)\, dt_0; \quad i = 1,\ 2,\ 3. \tag{2}$$

Here the expressions $J_i(t - t_0)$ satisfy the equation of heat conduction and certain initial and boundary conditions. Applying a Laplace transformation [2-3] to Eqs. (1) and (2)

$$f(p) = \int_0^\infty \exp(-pt)\, f(t)\, dt$$

and using the convolution theorem, we obtain

$$Q(p) = \sum_{i=1}^{n} q_i(p)\, s_i; \quad T(p) = \gamma_i(p)\, q_i(p). \tag{3}$$

Here $q_i(p)$ are complex thermal fluxes and $\gamma_i(p)$ are complex thermal potentials, $i = 1, 2, 3$. The solution of the system (3) in image space becomes

$$T(p) = Q(p) \left[\sum_{i=1}^{3} \frac{s_i}{\gamma_i(p)} \right]^{-1}; \quad Q_i(p) = T(p)\, \frac{s_i}{\gamma_i(p)}. \tag{4}$$

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The complex thermal potentials $\gamma_i(p)$ depend on the velocity of the heat sources, the geometry of the bodies and contact regions, and other factors. In order to obtain the solution in the original variables we have to take the resultant equations (4) and apply the Riemann–Mellin inversion formula [2-3]

$$f(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} f(p)\, e^{pt}\, dt.$$

The special importance of an operational solution of this kind lies in the possibility of deriving a variety of asymptotic expressions. From the general solution (4) it is easy to obtain the steady-state solution if the total quantity of heat evolved in unit time is constant $Q(t) = Q$; $Q(p) = Q/p$. According to the well-known limiting relationship $T(\infty) = \lim\limits_{p \to 0} pT(p)$ we obtain

$$T(\infty) = Q\left[\sum_{i=1}^{3} \frac{s_i}{\gamma_i(0)}\right]^{-1}; \quad Q_i(\infty) = T(\infty)\, \frac{s_i}{\gamma_i(0)}. \tag{5}$$

Here $\gamma_i(0)$ will be the real thermal potentials. In order to calculate the complex thermal potentials we use the method of heat sources.

It is well known that [2] in the half-space $z \geq 0$ an instantaneous heat source of strength q^* applied to the point $(x_0, y_0, 0)$ will, after the elapse of time $t - t_0$, induce the following temperature at the point (x, y, z):

$$\theta(t) = \frac{q^*(t_0)\, \omega}{4\lambda (\pi\omega)^{3/2} (t - t_0)} \exp\left[-\frac{\rho^2}{4\omega(t - t_0)}\right]. \tag{6}$$

Let a set of continuously acting heat sources $q(t_0)$, uniformly distributed over an area s of the boundary of the half-space, move at a constant velocity v parallel to the ox axis. Integrating Eq. (6) with respect to time t_0 and over the area s in the mobile coordinate system, we obtain

$$\theta(t) = \frac{\omega}{4\lambda(\pi\omega)^{3/2}} \iint\limits_{s} ds_0 \int\limits_{0}^{t} \frac{q(t_0)}{(t - t_0)^{3/2}} \exp\left[-\frac{\rho^2}{4\omega(t - t_0)} - \alpha^2\omega(t - t_0) - \alpha(x - x_0)\right] dt_0. \tag{7}$$

Applying a Laplace transformation and the convolution theorem, we have

$$\theta(p) = \frac{q(p)}{2\pi\lambda} \iint\limits_{s} \exp\left[-\alpha(x - x_0) - \rho\sqrt{\frac{p}{\omega} + \alpha^2}\right] \frac{1}{\rho}\, ds_0. \tag{8}$$

Let us calculate the complex thermal potentials for the case in which the heat sources are distributed over a circle or strip.

If the heat sources are distributed over a circle of radius R, and if in Eq. (8) we integrate with respect to φ_0 in the coordinates (r, φ, z), at the center of the circle $z = r = 0$ we shall have

$$\theta(p) = q(p)\, \gamma_k(p); \quad \gamma_k(p) = \frac{1}{\lambda} \int\limits_{0}^{R} I_0(\alpha r_0) \exp\left[-r_0 \sqrt{\frac{p}{\omega} + \alpha^2}\right] dr_0. \tag{9}$$

Using the Laplace method [4] with due allowance for the well-known expansion of the function $I_0(x)$ in power series, we may derive an asymptotic series for large p ($|p| \to \infty$; $\mathrm{Re}\, p > 0$),

$$\gamma_k(p) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(2n)!\, \alpha^{2n}}{(n!)^2\, 2^{2n}} \left(\frac{p}{\omega} + \alpha^2\right)^{-\frac{2n+1}{2}} \tag{10}$$

The steady-state temperature for the case of constant intensity $q(t) = q$ is given by the equation

$$\theta(\infty) = q\gamma_k(0); \quad \gamma_k(0) = \frac{1}{\lambda} \int\limits_{0}^{R} I_0(\alpha r_0) \exp(-r_0\alpha)\, dr_0. \tag{11}$$

With due allowance for the properties of Bessel functions the latter integral may be expressed in terms of the tabulated functions

$$\gamma_k(0) = \exp(-\alpha R)\,[I_0(\alpha R) + I_1(\alpha R)]. \tag{12}$$

If the heat sources are distributed over a band or strip $|x_0| \leq a$, by integrating Eq. (8) in Cartesian coordinates with respect to y_0 we obtain the following at points on the ox axis ($y = 0$):

$$\theta(p) = \frac{q(p)}{\lambda\pi} \int_{x-a}^{x+a} \exp(-\alpha u)\, K_0\left(|u| \sqrt{\frac{p}{\omega} + \alpha^2}\right) du. \quad \textbf{(13)}$$

In particular, in the center of the strip

$$\theta(p) = q(p)\,\gamma_{st}(p);\quad \gamma_{st}(p) = \frac{2}{\pi\lambda} \int_0^a \mathrm{ch}(\alpha u)\, K_0\left(u \sqrt{\frac{p}{\omega} + \alpha^2}\right) du. \quad \textbf{(14)}$$

In the present case it is more complicated to obtain an asymptotic expansion than to calculate the asymptotic series (10).

Remembering that the main contribution to the integral comes from the point $u = 0$, and discussing the situation as in the Laplace method, only replacing the exponential by the function $K_0(x)$, we obtain an asymptotic series for large p ($|p| \to \infty$, $\mathrm{Re}\, p > 0$)

$$\gamma_{st}(p) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{\alpha^{2n}(2n-1)!!}{(2n)!!} \left(\frac{p}{\omega} + \alpha^2\right)^{-\frac{2n+1}{2}}. \quad (15)$$

We note that for large p the first terms of the asymptotic series (10) and (15) coincide:

$$\gamma_{st}(p) = \gamma_k(p) = \left[\lambda \sqrt{\frac{p}{\omega} + \alpha^2}\right]^{-1}. \quad (16)$$

The steady-state temperature for the case $q(t) = q$ is written thus:

$$\theta(\infty) = q\gamma_{st}(0);\quad \gamma_{st}(0) = \frac{2}{\pi\lambda} \int_0^a \mathrm{ch}(\alpha u)\, K_0(\alpha u)\, du. \quad (17)$$

Using a method analogous to that used in deriving Eq. (12), we obtain an expression for the integral in terms of known functions

$$\gamma_{st}(0) = \frac{2a}{\pi\lambda}\,[\mathrm{ch}(\alpha a)\, K_0(\alpha a) + \mathrm{sh}(\alpha a)\, K_1(\alpha a)]. \quad (18)$$

According to the general scheme we express the complex contact temperature T(p) for each body in terms of the complex thermal fluxes $q_i(p)$.

In relation to the cutter, which is regarded as an octant (one eight of space), the heat sources are stationary and are distributed over its front and back faces in the form of quarters of circles with radii $r_f = 2\sqrt{bl_f/\pi}$; $r_b = \sqrt{bl_b/\pi}$ and a total area of $s_1 = bl$. Allowing for the superposition principle and using Eq. (9) with $\alpha = 0$, we obtain the complex temperature at the tip of the cutter in the form

$$T(p) = q_1(p)\,\gamma_1(p), \quad (19)$$

$$\gamma_1(p) = \frac{1}{\lambda} \sqrt{\frac{\omega_1}{p}} \left[2 - \exp\left(-r_f \sqrt{\frac{p}{\omega_1}}\right) - \exp\left(-r_b \sqrt{\frac{p}{\omega_1}}\right)\right];\quad \gamma_1(0) = \frac{r_f + r_b}{\lambda_1}. \quad (20)$$

We regard the work-piece as a half-space; along its surface, traveling at a velocity $v_2 = v$, are heat sources distributed in the form of rectangles $|x_0| \leq a_2$; $|y_0| \leq b/2$ with areas $s_2 = bl_2$; $l_2 = l_b + l_d$; $a_2 = l_2/2$. In order to determine the temperature we supplement the heat sources so as to form a strip $-\infty < y_0 < \infty$, the characteristic dimension in the direction of motion of the sources being preserved.

We take the chip or shaving as a half-plate of width b; over its end surface, traveling at a velocity $v_3 = v/k$, are heat sources distributed over the range $|x_0| < l_b/2 = a_3$ with areas $s_3 = bl_b$; $l_3 = l_b + l_d$. On the foregoing assumptions, the contact temperature of the work-piece and shaving (the temperature in the center of the source distribution) is determined by Eq. (14), and may be written as follows:

$$T(p) = \gamma_i(p)\, q_i(p);\quad \gamma_i(p) = \frac{2}{\pi\lambda_i} \int_0^{a_i} K_0\left(u \sqrt{\frac{p}{\omega_i} + \alpha_i^2}\right) \mathrm{ch}(\alpha_i u)\, du, \quad (21)$$

$$\gamma_i(0) = \frac{2a_i}{\pi\lambda_i} [\mathrm{ch}(\alpha_i a) K_0(\alpha_i a) + \mathrm{sh}(\alpha_i a) K_1(\alpha_i a_i)];$$
$$\alpha_i = \frac{v_i}{2\omega_i}; \quad i = 2, 3. \tag{22}$$

Substituting the resultant values of the complex thermal potentials (21) and (20) into (4), we obtain a solution to the problem in image space. Equations (5), with due allowance for the values of the thermal potentials at $p = 0$, taken together with (22) and (20), give the steady-state contact temperature and the stable thermal fluxes.

Let us study the resultant solution after long and short periods of time t for the case in which the total amount of heat liberated in the cutting zone per unit time constant: $Q(t) = Q$; $Q(p) = Q/p$. For large values of p, on allowing for the asymptotic equation (16), we obtain

$$T(p) = \frac{Q}{p} \left[\sum_{i=1}^{3} \varepsilon_i \lambda_i s_i \sqrt{\frac{p}{\omega_i} + \alpha_i^2} \right]^{-1}. \tag{23}$$

Here for convenience of writing we have taken $\alpha_1 = 0$; $\varepsilon_1 = 2$; $\varepsilon_2 = \varepsilon_3 = 1$. Expanding the latter expression in series in powers of $1/\sqrt{p}$ and confining attention to two of the terms we obtain

$$T = \frac{Q}{p} \left\{ \frac{1}{M\sqrt{p}} - \frac{N}{2M^2} \frac{1}{p} \right\}, \tag{24}$$

where $M = \sum_{i=1}^{3} \varepsilon_i \lambda_i s_i (1/\sqrt{\omega_i})$ is the total conductivity of the system; $N = \sum_{i=1}^{3} \varepsilon_i \lambda_i s_i \alpha_i^2 \sqrt{\omega_i}$. In an analogous way we obtain expansions for the thermal fluxes

$$Q_i(p) = \frac{Q\lambda_i s_i \varepsilon_i}{pM\sqrt{\omega_i}} \left\{ 1 + \frac{1}{2p} \left(\alpha_i^2 \omega_i - \frac{N}{M} \right) \right\}, \quad i = 1, 2, 3. \tag{25}$$

Transforming to the original variables, we shall have

$$T(t) = \frac{2Q\sqrt{t}}{M\sqrt{\pi}} \left(1 - \frac{N}{3M} t \right),$$
$$Q_i(t) = \frac{\varepsilon_i \lambda_i s_i Q}{M\sqrt{\omega_i}} \left\{ 1 + \frac{t}{2} \left(\alpha_i^2 \omega_i - \frac{N}{M} \right) \right\}, \quad i = 1, 2, 3. \tag{26}$$

It follows from the latter equations that at the initial instant the thermal fluxes are distributed in the ratio

$$Q_1 : Q_2 : Q_3 = \frac{2\lambda_1 s_1}{\sqrt{\omega_1}} : \frac{\lambda_2 s_2}{\sqrt{\omega_2}} : \frac{\lambda_3 s_3}{\sqrt{\omega_3}}.$$

Let us study the solution for small p (large t), basing our considerations on the theorem arbitrarily known as "the rule of fractional indices" [3].

If the image f(p) may be expanded in the neighborhood of the point p_0 in the power series

$$f(p) = \sum_{n=0}^{\infty} c_n (p - p_0)^{\lambda_n}; \quad \lambda_0 < \lambda_1 < \ldots \to \infty, \tag{27}$$

the asymptotic expansion of the original will, as $t \to \infty$, form a series

$$f(t) = e^{p_0 t} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\lambda_n)} t^{-\lambda_n - 1}. \tag{28}$$

Here we must put $[\Gamma(-\lambda_n)]^{-1} = 0$ when λ_n takes a positive whole-number value.

We may express the contact temperature (4) in the neighborhood of zero [with due allowance for Eqs. (20) and (21)] in the form of a series

$$T(p) = \frac{1}{p} \sum_{n=0}^{\infty} c_n p^{n/2}. \tag{29}$$

In order to determine the first two coefficients of this series we may consider the following expressions:

$$c_0 = \lim_{p \to 0} [pT(p)]; \quad c_1 = \lim_{p \to 0} \{2\sqrt{p}\,[pT(p)]'\}. \tag{30}$$

The coefficient c_0, equal to the steady contact temperature, was calculated earlier in Eq. (5).

For the second coefficient c_1, after executing the limiting transition we have

$$c_1 = -\frac{\lambda_1 s_1 c_0^2 \Delta}{2Q\sqrt{\omega_1}}; \quad \Delta = \frac{r_b^2 + r_f^2}{(r_f + r_b)^2}.$$

According to Eq. (28) we may write the contact temperature as follows, confining attention to only two terms in the series:

$$T(t) = T(\infty)\left[1 - \frac{\lambda_1 s_1 \Delta}{2\sqrt{\pi\omega_1 t}\,Q}\, T(\infty)\right].$$

On the basis of this last equation we may conclude that, as the thermal conductivity of the cutter and the contact area are reduced, the stabilization time of the process diminishes.

The resultant solutions may find applications in other processes as well as cutting. Thus, by putting $\lambda_3 = 0$ we obtain a solution to the problem of transient heat transfer between two bodies. Heat transfer in a system of two bodies arises in finishing and strengthening processes, in problems relating to friction between two bodies, and so forth.

NOTATION

ω, λ, thermal conductivity and thermal diffusivity; v, velocity of heat sources (cutting speed); $I_i(x)$, $K_i(x)$, modified Bessel functions of the first and second kinds; b, width of cut; l_f, l_b, lengths of contacts along the front and back faces of the tool; l_d, length of deformation area, $l = l_f + l_b$; k, shrinkage coefficient of shaving; $\rho^2 = (x - x_0)^2 + (y - y_0)^2 + z^2$; $\alpha = v/2\omega$.

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